Topics of today:

Singular DCJ-indel distance and sorting:

1. Review
2. Capped relational graph of canonical genomes
3. Capped relational graph of singular genomes
4. Indel-potential of cycles via transitions

Components of a relational graph

Cycle with k extremity-edges: k-cycle or c_k

Path with k extremity-edges: k-path or p_k

if $k = 0$ the component is a **singleton**

Disjoint sets of components: $$\begin{cases} \mathcal{C} = \{c_k : k \geq 2\} : \text{set of cycles } (k \text{ is even}) \\ \mathcal{S} = \{c_k : k = 0\} : \text{set of circular singletons} \\ \mathcal{P}_{\mathbb{A}\mathbb{A}} = \{p_k : \text{starts and ends in } \mathbb{A}\} \ : \\ \qquad\qquad\qquad \text{set of } \mathbb{A}\mathbb{A}\text{-paths } (k \geq 0 \text{ is even}) \\ \mathcal{P}_{\mathbb{B}\mathbb{B}} = \{p_k : \text{starts and ends in } \mathbb{B}\} \ : \\ \qquad\qquad\qquad \text{set of } \mathbb{B}\mathbb{B}\text{-paths } (k \geq 0 \text{ is even}) \\ \mathcal{P}_{\mathbb{A}\mathbb{B}} = \{p_k : \text{starts in } \mathbb{A} \text{ and ends in } \mathbb{B}\} \ : \\ \qquad\qquad\qquad \text{set of } \mathbb{A}\mathbb{B}\text{-paths } (k \geq 1 \text{ is odd}) \end{cases}$$

DCJ-sorted (or short) components: 2-cycles and 1-paths (and 0-cycles and 0-paths)

Long components: k-cycles (with $k \geq 4$) and k-paths (with $k \geq 2$)

DCJ-sorting a long component C: transforming C into a set of DCJ-sorted components with DCJ-operations

Types of DCJ operation

With respect to the position of the cuts:

Internal: either a single-cut operation or two cuts applied in the same component

Recombination: each cut is applied in a distinct component

With respect to the effect on the relational graph:

Gaining: creates one cycle or two $\mathbb{AB}$-paths
$\Delta_{\text{DCJ}} = 0$

Neutral: preserves the number of cycles and of $\mathbb{AB}$-paths
$\Delta_{\text{DCJ}} = 1$

Losing: destroys one cycle or two $\mathbb{AB}$-paths
$\Delta_{\text{DCJ}} = 2$

Each component can be sorted separately...

...with an internal gaining DCJ at each step:

Cycle: creates a new cycle at each step

$\mathbb{A}\mathbb{B}$-path: creates a new cycle at each step

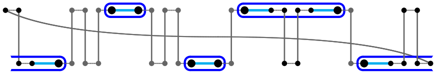
$\mathbb{A}\mathbb{A}$-path: creates a new cycle at each step, eventually one step is a single cut (on $\mathbb{B}$) that creates two $\mathbb{A}\mathbb{B}$-paths

$\mathbb{B}\mathbb{B}$-path: analogous to $\mathbb{A}\mathbb{A}$-path

Accummulating runs

indel edges , runs

One indel-enclosing cycle:



$\Lambda = 4$

Each **run** can be **accumulated** with internal gaining DCJ operations and then inserted/deleted at once

$\Rightarrow$ Second upper bound:

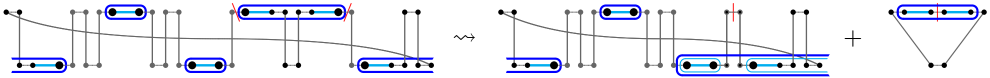
$$\mathbf{d}_{\mathrm{DCJ}}^{\mathrm{ID}}(\mathbb{A},\mathbb{B}) \leq n - |\mathcal{C}| - \frac{|\mathcal{P}_{\mathbb{A}\mathbb{B}}|}{2} + \sum_{C \in RG(\mathbb{A},\mathbb{B})} \Lambda(C)$$

Merging runs with internal gaining DCJ operations

DCJ operations can modify the number of runs by at most two:

$$\text{A DCJ operation can have} \begin{cases} \Delta_\Lambda = -2 & \text{(merges two pairs of runs)} \\ \Delta_\Lambda = -1 & \text{(merges one pair of runs)} \\ \Delta_\Lambda = 0 & \text{(preserves the runs)} \\ \Delta_\Lambda = 1 & \text{(splits one run)} \\ \Delta_\Lambda = 2 & \text{(splits two runs)} \end{cases}$$

A **gaining DCJ operation** applied to two adjacency-edges belonging to the same indel-enclosing component can **decrease** the number of runs:



$\Lambda = 4$	$\rightsquigarrow$	2	$+$	$1 = 3$ $(\Delta_\Lambda{=}{-1})$

Indel-potential $\lambda(C)$ of a component C:

minimum number of runs that we can obtain by DCJ-sorting C with internal gaining DCJ operations

Indel-potential of a cycle C - with $\Lambda(C) = 0, 1, 2, 4, 6, 8, \ldots$

We will show that $\lambda(C)$ depends only on the value $\Lambda(C)$: denote $\lambda(C) = \lambda(\Lambda(C))$

$$\Lambda(C) = 0 \Rightarrow \lambda(0) = 0$$
$$\Lambda(C) = 1 \Rightarrow \lambda(1) = 1$$
$$\Lambda(C) = 2 \Rightarrow \lambda(2) = 2$$
$$\Lambda(C) = 4 \Rightarrow \lambda(4) = 3 \text{ (can be verified by listing all cases)}$$

$\Lambda(C) \geq 6$: extract 3 runs from C into a new cycle $\rightarrow$ garantees that $\Delta_\Lambda = -2$

$$\text{two resulting cycles: } \begin{cases} \text{one with 2 runs} \\ \text{one with } \Lambda(C) - 4 \text{ runs} \end{cases}$$

$$\Rightarrow \lambda(6) = \lambda(2) + \lambda(2) = 2 + 2 = 4$$
$$\Rightarrow \lambda(8) = \lambda(2) + \lambda(4) = 2 + 3 = 5$$
$$\Rightarrow \lambda(10) = \lambda(2) + \lambda(6) = 2 + 4 = 6$$

Λ	λ
0	0
1	1
2	2
4	3
6	4
8	5
$\vdots$	$\vdots$

$$\text{Induction: } \begin{cases} \text{hypothesis: } \lambda(\Lambda(C)) = \frac{\Lambda(C)}{2} + 1 \\ \text{base cases: } \lambda(1) = 1,\ \lambda(2) = 2 \text{ and } \lambda(4) = 3 \end{cases}$$

Induction step: in general, for $\Lambda(C) \geq 6$, we can state

$$\begin{aligned} \lambda(\Lambda(C)) &= \lambda(2) + \lambda(\Lambda(C) - 4) \\ &= 2 + \left(\frac{\Lambda(C) - 4}{2} + 1\right) \\ &= \frac{\Lambda(C)}{2} + 1 \end{aligned}$$

Indel-potential λ of a path P - with $\Lambda(P) = 0, 1, 2, 3, 4, 5, 6, 7, 8, \ldots$

Since $\lambda(P)$ depends only on the value $\Lambda(P)$, we can denote $\lambda(P) = \lambda(\Lambda(P))$

$$\Lambda(P) = 0 \Rightarrow \lambda(0) = 0$$

$$\Lambda(P) = 1 \Rightarrow \lambda(1) = 1$$

$$\Lambda(P) = 2 \Rightarrow \lambda(2) = 2$$

$$\Lambda(P) \geq 3 : \begin{cases} \text{if } \Lambda(P) \text{ is even, then } \lambda(\Lambda(P)) = \frac{\Lambda(P)}{2} + 1 \\ \text{else } \lambda(\Lambda(P)) = \lambda(\Lambda(P) - 1) \end{cases}$$

Λ	λ
0	0
1	1
2 3	2 2
4 5	3 3
6 7	4 4
$\vdots$	$\vdots$

In general, for $\Lambda(P) \geq 2$, we have

$$\lambda(\Lambda(P)) = \left\lceil \frac{\Lambda(P) + 1}{2} \right\rceil$$

Indel-potential λ of a component C

If C is a singleton: $\lambda(C) = 1$

If C is a cycle:

$$\lambda(C) = \begin{cases} 0 & \text{if } \Lambda(C) = 0 \ (C \text{ is indel-free}) \\ 1 & \text{if } \Lambda(C) = 1 \\ \frac{\Lambda(C)}{2} + 1 & \text{if } \Lambda(C) \geq 2 \end{cases}$$

If C is a path:

$$\lambda(C) = \begin{cases} 0 & \text{if } \Lambda(C) = 0 \ (C \text{ is indel-free}) \\ \left\lceil \frac{\Lambda(C)+1}{2} \right\rceil & \text{if } \Lambda(C) \geq 1 \end{cases}$$

Λ	λ	
0	0	paths and cycles
1	1	paths, cycles and singletons
2	2	paths and cycles
3	2	paths
4	3	paths and cycles
5	3	paths
6	4	paths and cycles
7	4	paths
$\vdots$	$\vdots$	

In general, for any component C:

$$\lambda(C) = \begin{cases} 0 & \text{if } \Lambda(C) = 0 \ (C \text{ is indel-free}) \\ \left\lceil \frac{\Lambda(C)+1}{2} \right\rceil & \text{if } \Lambda(C) \geq 1 \end{cases}$$

Third upper bound: $$d_{\mathrm{DCJ}}^{\mathrm{ID}}(\mathbb{A}, \mathbb{B}) \leq n - |\mathcal{C}| - \frac{|\mathcal{P}_{\mathbb{AB}}|}{2} + \sum_{C \in RG} \lambda(C)$$

(gaining DCJ operations + indels sorting components separately)

Effect of a DCJ operation on the third upper bound:

DCJ-types of DCJ operation $\begin{cases} \Delta_{\rm DCJ} = 0 \textbf{ (gaining):} & \text{creates one cycle or two } \mathbb{AB}\text{-paths} \\ \Delta_{\rm DCJ} = 1 \textbf{ (neutral):} & \text{preserves the numbers of cycles and of } \mathbb{AB}\text{-paths} \\ \Delta_{\rm DCJ} = 2 \textbf{ (losing):} & \text{destroys one cycle or two } \mathbb{AB}\text{-paths} \end{cases}$

Indel-types of DCJ operation $\begin{cases} \Delta_\lambda = -2 & : \text{decreases the overall indel-potential by two} \\ \Delta_\lambda = -1 & : \text{decreases the overall indel-potential by one} \\ \Delta_\lambda = 0 & : \text{does not change the overall indel-potential} \\ \Delta_\lambda = 1 & : \text{increases the overall indel-potential by one} \\ \Delta_\lambda = 2 & : \text{increases the overall indel-potential by two} \end{cases}$

Effect of a DCJ operation ρ on the third upper bound: $\Delta^{\lambda}_{\rm DCJ}(\rho) = \Delta_{\rm DCJ}(\rho) + \Delta_\lambda(\rho)$

DCJ Operations that can decrease the third upper bound: $\begin{cases} \Delta_{\rm DCJ} = 0 \text{ (gaining) and } \Delta_\lambda = -2 & : \ \Delta^{\lambda}_{\rm DCJ} = -2 \\ \Delta_{\rm DCJ} = 0 \text{ (gaining) and } \Delta_\lambda = -1 & : \ \Delta^{\lambda}_{\rm DCJ} = -1 \\ \Delta_{\rm DCJ} = 1 \text{ (neutral) and } \Delta_\lambda = -2 & : \ \Delta^{\lambda}_{\rm DCJ} = -1 \end{cases}$

- By definition: any internal gaining DCJ operation ρ (applied to a single component) has $\Delta_\lambda(\rho) \geq 0$ and, consequently, $\Delta^{\lambda}_{\rm DCJ}(\rho) \geq 0$
- Any losing DCJ operation ρ has $\Delta^{\lambda}_{\rm DCJ}(\rho) \geq 0$

DCJ operations involving cycles

Λ	λ
0	0
1	1
2	2
4	3
6	4
8	5
$\vdots$	$\vdots$

- Any recombination involving two cycles is losing and has $\Delta^{\lambda}_{\text{DCJ}} \geq 0$ (cannot decrease the DCJ-indel distance)

- An internal DCJ operation ρ applied to a cycle C can be:
  - Gaining, with $\Delta^{\lambda}_{\text{DCJ}}(\rho) \geq 0$ (cannot decrease the DCJ-indel distance)
  - Neutral ($\Delta_{\text{DCJ}}(\rho) = 1$):

    If $\Lambda(C) \geq 4$, the DCJ ρ can merge at most two pairs of runs: $\Delta_{\Lambda}(\rho) \geq -2$ and $\Delta_{\lambda}(\rho) \geq -1$

    $\Rightarrow$ Any internal neutral DCJ operation applied to a cycle has $\Delta^{\lambda}_{\text{DCJ}} \geq 0$ (cannot decrease the DCJ-indel distance)

If singular genomes $\mathbb{A}$ and $\mathbb{B}$ are circular, the graph $RG(\mathbb{A}, \mathbb{B})$ has only cycles (and eventually singletons).

In this case:

$$\mathrm{d}^{\text{ID}}_{\text{DCJ}}(\mathbb{A}, \mathbb{B}) = n - |\mathcal{C}| + \sum_{C \in RG} \lambda(C)$$

DCJ operations involving paths

Λ	λ
0	0
1	1
2 3	2 2
4 5	3 3
6 7	4 4
$\vdots$	$\vdots$

- Any recombination involving a path and a cycle is losing and has $\Delta^{\lambda}_{\mathrm{DCJ}} \geq 0$ (cannot decrease the DCJ-indel distance)

- An internal DCJ operation ρ applied to a path P can be:
  - Gaining, with $\Delta^{\lambda}_{\mathrm{DCJ}}(\rho) \geq 0$ (cannot decrease the DCJ-indel distance)
  - Neutral ($\Delta_{\mathrm{DCJ}}(\rho) = 1$):

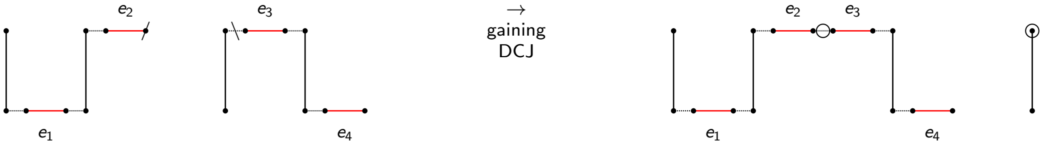
    If $\Lambda(P) \geq 4$, the DCJ ρ can merge at most two pairs of runs: $\Delta_{\Lambda}(\rho) \geq -2$ and $\Delta_{\lambda}(\rho) \geq -1$

    $\Rightarrow$ Any internal neutral DCJ operation applied to a path has $\Delta^{\lambda}_{\mathrm{DCJ}} \geq 0$ (cannot decrease the DCJ-indel distance)

Path recombinations can have $\Delta^{\lambda}_{\mathrm{DCJ}} \leq -1$

A gaining (**deducting**) path recombination with $\Delta^{\lambda}_{\mathrm{DCJ}} = -2$:

Sources			**Resultants**		
$(\sum \lambda = 2 + 2 = 4)$			$(\sum \lambda = 2 + 0 = 2)$		
$\mathbb{AA}$	$+$	$\mathbb{BB}$	$\mathbb{AB}$	$+$	$\mathbb{AB}$
2 runs	$+$	2 runs	3 runs	$+$	no run



$$\mathbb{AA}_{\mathcal{BA}} + \mathbb{BB}_{\mathcal{AB}} = \begin{cases} \mathbb{AB}_{\mathcal{BAB}} + \mathbb{AB}_{\varepsilon} \\ (\mathbb{AB}_{\mathcal{ABA}} + \mathbb{AB}_{\varepsilon}) \\ (\mathbb{AB}_{\mathcal{A}} \quad + \mathbb{AB}_{\mathcal{B}}) \end{cases} \quad \text{(all variants have } \Delta^{\lambda}_{\mathrm{DCJ}} = -2\text{)}$$

Deducting path recombinations

have $\Delta^{\lambda}_{\mathrm{DCJ}} \leq -1$

General DCJ-indel distance formula:

$$d^{\mathrm{ID}}_{\mathrm{DCJ}}(\mathbb{A}, \mathbb{B}) = n - |\mathcal{C}| - \frac{|\mathcal{P}_{\mathbb{AB}}|}{2} + \sum_{C \in RG} \lambda(C) - \delta,$$

where δ is the value obtained by optimizing deducting path recombinations

Optimizing deducting path recombinations (for computing δ)

Run-type of a path $\begin{cases} \varepsilon & \equiv & \varepsilon & \text{(empty)} \\ \mathcal{ABAB}\ldots\mathcal{A} & \equiv & \mathcal{A} & \text{(odd)} \\ \mathcal{BABA}\ldots\mathcal{B} & \equiv & \mathcal{B} & \text{(odd)} \\ \mathcal{ABAB}\ldots\mathcal{AB} & \equiv & \mathcal{AB} & \text{(even)} \\ \mathcal{BABA}\ldots\mathcal{BA} & \equiv & \mathcal{BA} & \text{(even)} \end{cases}$

Path types $\begin{cases} \mathbb{AA}_\varepsilon, \mathbb{AA}_\mathcal{A}, \mathbb{AA}_\mathcal{B}, \mathbb{AA}_\mathcal{AB} (\equiv \mathbb{AA}_\mathcal{BA}) \\ \mathbb{BB}_\varepsilon,\ \mathbb{BB}_\mathcal{A},\ \mathbb{BB}_\mathcal{B}, \mathbb{BB}_\mathcal{AB} (\equiv \mathbb{BB}_\mathcal{BA}) \\ \mathbb{AB}_\varepsilon,\ \mathbb{AB}_\mathcal{A},\ \mathbb{AB}_\mathcal{B},\ \mathbb{AB}_\mathcal{AB},\ \mathbb{AB}_\mathcal{BA} \\ \quad \Rightarrow \text{an } \mathbb{AB}\text{-path is always read from } \mathbb{A} \text{ to } \mathbb{B} \end{cases}$

Deducting path recombinations that allow the best reuse of the resultants:

sources	resultants	Δ_λ	Δ_{DCJ}	$\Delta^\lambda_{\mathrm{DCJ}}$
$\mathbb{AA}_\mathcal{AB} + \mathbb{BB}_\mathcal{AB}$	$\bullet + \bullet$	-2	0	-2
$\mathbb{AA}_\mathcal{AB} + \mathbb{BB}_\mathcal{A}$	$\bullet + \mathbb{AB}_\mathcal{BA}$	-1	0	-1
$\mathbb{AA}_\mathcal{AB} + \mathbb{BB}_\mathcal{B}$	$\bullet + \mathbb{AB}_\mathcal{AB}$	-1	0	-1
$\mathbb{AA}_\mathcal{A} + \mathbb{BB}_\mathcal{AB}$	$\bullet + \mathbb{AB}_\mathcal{AB}$	-1	0	-1
$\mathbb{AA}_\mathcal{B} + \mathbb{BB}_\mathcal{AB}$	$\bullet + \mathbb{AB}_\mathcal{BA}$	-1	0	-1
$\mathbb{AA}_\mathcal{A} + \mathbb{BB}_\mathcal{A}$	$\bullet + \bullet$	-1	0	-1
$\mathbb{AA}_\mathcal{B} + \mathbb{BB}_\mathcal{B}$	$\bullet + \bullet$	-1	0	-1

sources	resultants	Δ_λ	Δ_{DCJ}	$\Delta^\lambda_{\mathrm{DCJ}}$
$\mathbb{AA}_\mathcal{AB} + \mathbb{AA}_\mathcal{AB}$	$\mathbb{AA}_\mathcal{A} + \mathbb{AA}_\mathcal{B}$	-2	$+1$	-1
$\mathbb{BB}_\mathcal{AB} + \mathbb{BB}_\mathcal{AB}$	$\mathbb{BB}_\mathcal{A} + \mathbb{BB}_\mathcal{B}$	-2	$+1$	-1
$\mathbb{AA}_\mathcal{AB} + \mathbb{AB}_\mathcal{AB}$	$\bullet + \mathbb{AA}_\mathcal{A}$	-2	$+1$	-1
$\mathbb{AA}_\mathcal{AB} + \mathbb{AB}_\mathcal{BA}$	$\bullet + \mathbb{AA}_\mathcal{B}$	-2	$+1$	-1
$\mathbb{BB}_\mathcal{AB} + \mathbb{AB}_\mathcal{AB}$	$\bullet + \mathbb{BB}_\mathcal{B}$	-2	$+1$	-1
$\mathbb{BB}_\mathcal{AB} + \mathbb{AB}_\mathcal{BA}$	$\bullet + \mathbb{BB}_\mathcal{A}$	-2	$+1$	-1
$\mathbb{AB}_\mathcal{AB} + \mathbb{AB}_\mathcal{BA}$	$\bullet + \bullet$	-2	$+1$	-1

$\bullet \begin{cases} \mathbb{AB}_\varepsilon \\ \mathbb{AB}_\mathcal{A} \\ \mathbb{AB}_\mathcal{B} \end{cases}$

Path recombinations with $\Delta^\lambda_{\mathrm{DCJ}} = 0$ creating resultants that can be used in deducting recombinations:

sources	resultants	Δ_λ	Δ_{DCJ}	$\Delta^\lambda_{\mathrm{DCJ}}$
$\mathbb{AA}_\mathcal{A} + \mathbb{AB}_\mathcal{BA}$	$\bullet + \mathbb{AA}_\mathcal{AB}$	-1	$+1$	0
$\mathbb{AA}_\mathcal{B} + \mathbb{AB}_\mathcal{AB}$	$\bullet + \mathbb{AA}_\mathcal{AB}$	-1	$+1$	0
$\mathbb{BB}_\mathcal{A} + \mathbb{AB}_\mathcal{AB}$	$\bullet + \mathbb{BB}_\mathcal{AB}$	-1	$+1$	0
$\mathbb{BB}_\mathcal{B} + \mathbb{AB}_\mathcal{BA}$	$\bullet + \mathbb{BB}_\mathcal{AB}$	-1	$+1$	0

sources	resultants	Δ_λ	Δ_{DCJ}	$\Delta^\lambda_{\mathrm{DCJ}}$
$\mathbb{AA}_\mathcal{A} + \mathbb{BB}_\mathcal{B}$	$\bullet + \mathbb{AB}_\mathcal{AB}$	0	0	0
$\mathbb{AA}_\mathcal{B} + \mathbb{BB}_\mathcal{A}$	$\bullet + \mathbb{AB}_\mathcal{BA}$	0	0	0
$\mathbb{AB}_\mathcal{AB} + \mathbb{AB}_\mathcal{AB}$	$\mathbb{AA}_\mathcal{A} + \mathbb{BB}_\mathcal{B}$	-2	$+2$	0
$\mathbb{AB}_\mathcal{BA} + \mathbb{AB}_\mathcal{BA}$	$\mathbb{AA}_\mathcal{B} + \mathbb{BB}_\mathcal{A}$	-2	$+2$	0

Sources:

$\mathbb{AA}_\mathcal{AB}$: $\mathtt{W}$

$\mathbb{AA}_\mathcal{A}$: $\overline{\mathtt{W}}$

$\mathbb{AA}_\mathcal{B}$: $\underline{\mathtt{W}}$

$\mathbb{BB}_\mathcal{AB}$: $\mathtt{M}$

$\mathbb{BB}_\mathcal{A}$: $\overline{\mathtt{M}}$

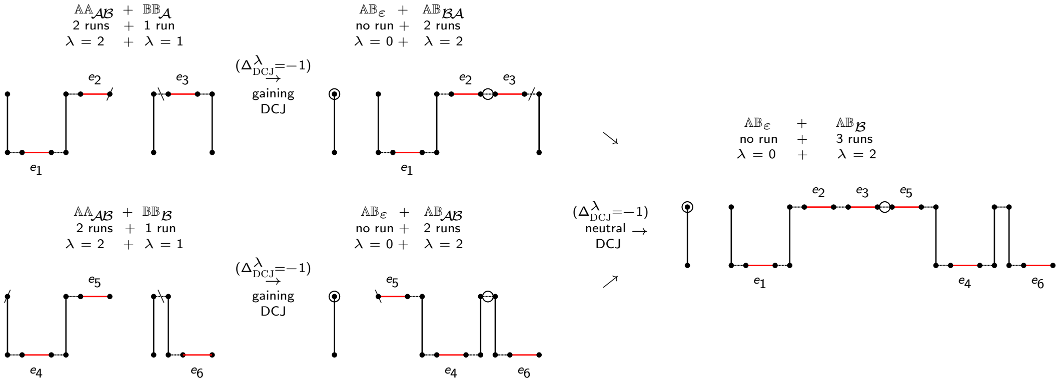
$\mathbb{BB}_\mathcal{B}$: $\underline{\mathtt{M}}$

$\mathbb{AB}_\mathcal{AB}$: $\mathtt{Z}$

$\mathbb{AB}_\mathcal{BA}$: $\mathtt{N}$

Optimizing deducting path recombinations (for computing δ)

$$\text{Deducting chain of path recombinations} \begin{cases} \text{transforming} & 2 \times \mathbb{AA}_{\mathcal{AB}} + \mathbb{BB}_{\mathcal{A}} + \mathbb{BB}_{\mathcal{B}} \\ \text{into} & 3 \times \mathbb{AB}_{\varepsilon} + \mathbb{AB}_{\mathcal{B}} \\ \text{with} & \text{overall } \Delta^{\lambda}_{\mathrm{DCJ}} = -3 \end{cases}$$



	id	sources			resultants				$\mathbf{\Delta}^{\boldsymbol{\lambda}}_{\text{DCJ}}$	scr
$\mathcal{P}$	$\mathtt{WM}$	$\mathbb{AA}_{\mathcal{AB}}$	$\mathbb{BB}_{\mathcal{AB}}$	—	—	—	—	$2\times\bullet$	-2	-1
$\mathcal{Q}$	$\mathtt{WW\overline{M}\underline{M}}$	$2\times\mathbb{AA}_{\mathcal{AB}}$	$\mathbb{BB}_{\mathcal{A}}+\mathbb{BB}_{\mathcal{B}}$	—	—	—	—	$4\times\bullet$	-3	$-3/4$
	$\mathtt{MM\overline{W}\underline{W}}$	$\mathbb{AA}_{\mathcal{A}}+\mathbb{AA}_{\mathcal{B}}$	$2\times\mathbb{BB}_{\mathcal{AB}}$	—	—	—	—	$4\times\bullet$	-3	$-3/4$
$\mathcal{T}$	$\mathtt{WZ\overline{M}}$	$\mathbb{AA}_{\mathcal{AB}}$	$\mathbb{BB}_{\mathcal{A}}$	$\mathbb{AB}_{\mathcal{AB}}$	—	—	—	$3\times\bullet$	-2	$-2/3$
	$\mathtt{WW\overline{M}}$	$2\times\mathbb{AA}_{\mathcal{AB}}$	$\mathbb{BB}_{\mathcal{A}}$	—	$\mathbb{AA}_{\mathcal{B}}$	—	—	$2\times\bullet$	-2	$-2/3$
	$\mathtt{WN\underline{M}}$	$\mathbb{AA}_{\mathcal{AB}}$	$\mathbb{BB}_{\mathcal{B}}$	$\mathbb{AB}_{\mathcal{BA}}$	—	—	—	$3\times\bullet$	-2	$-2/3$
	$\mathtt{WW\underline{M}}$	$2\times\mathbb{AA}_{\mathcal{AB}}$	$\mathbb{BB}_{\mathcal{B}}$	—	$\mathbb{AA}_{\mathcal{A}}$	—	—	$2\times\bullet$	-2	$-2/3$
	$\mathtt{MN\overline{W}}$	$\mathbb{AA}_{\mathcal{A}}$	$\mathbb{BB}_{\mathcal{AB}}$	$\mathbb{AB}_{\mathcal{BA}}$	—	—	—	$3\times\bullet$	-2	$-2/3$
	$\mathtt{MM\overline{W}}$	$\mathbb{AA}_{\mathcal{A}}$	$2\times\mathbb{BB}_{\mathcal{AB}}$	—	—	$\mathbb{BB}_{\mathcal{B}}$	—	$2\times\bullet$	-2	$-2/3$
	$\mathtt{MZ\underline{W}}$	$\mathbb{AA}_{\mathcal{B}}$	$\mathbb{BB}_{\mathcal{AB}}$	$\mathbb{AB}_{\mathcal{AB}}$	—	—	—	$3\times\bullet$	-2	$-2/3$
	$\mathtt{MM\underline{W}}$	$\mathbb{AA}_{\mathcal{B}}$	$2\times\mathbb{BB}_{\mathcal{AB}}$	—	—	$\mathbb{BB}_{\mathcal{A}}$	—	$2\times\bullet$	-2	$-2/3$
$\mathcal{S}$	$\mathtt{ZN}$	—	—	$\mathbb{AB}_{\mathcal{AB}}+\mathbb{AB}_{\mathcal{BA}}$	—	—	—	$2\times\bullet$	-1	$-1/2$
	$\overline{\mathtt{WM}}$	$\mathbb{AA}_{\mathcal{A}}$	$\mathbb{BB}_{\mathcal{A}}$	—	—	—	—	$2\times\bullet$	-1	$-1/2$
	$\underline{\mathtt{WM}}$	$\mathbb{AA}_{\mathcal{B}}$	$\mathbb{BB}_{\mathcal{B}}$	—	—	—	—	$2\times\bullet$	-1	$-1/2$
	$\mathtt{W\overline{M}}$	$\mathbb{AA}_{\mathcal{AB}}$	$\mathbb{BB}_{\mathcal{A}}$	—	—	—	$\mathbb{AB}_{\mathcal{BA}}$	$\bullet$	-1	$-1/2$
	$\mathtt{W\underline{M}}$	$\mathbb{AA}_{\mathcal{AB}}$	$\mathbb{BB}_{\mathcal{B}}$	—	—	—	$\mathbb{AB}_{\mathcal{AB}}$	$\bullet$	-1	$-1/2$
	$\mathtt{WZ}$	$\mathbb{AA}_{\mathcal{AB}}$	—	$\mathbb{AB}_{\mathcal{AB}}$	$\mathbb{AA}_{\mathcal{A}}$	—	—	$\bullet$	-1	$-1/2$
	$\mathtt{WN}$	$\mathbb{AA}_{\mathcal{AB}}$	—	$\mathbb{AB}_{\mathcal{BA}}$	$\mathbb{AA}_{\mathcal{B}}$	—	—	$\bullet$	-1	$-1/2$
	$\mathtt{WW}$	$2\times\mathbb{AA}_{\mathcal{AB}}$	—	—	$\mathbb{AA}_{\mathcal{A}}+\mathbb{AA}_{\mathcal{B}}$	—	—	—	-1	$-1/2$
	$\mathtt{M\overline{W}}$	$\mathbb{AA}_{\mathcal{A}}$	$\mathbb{BB}_{\mathcal{AB}}$	—	—	—	$\mathbb{AB}_{\mathcal{AB}}$	$\bullet$	-1	$-1/2$
	$\mathtt{M\underline{W}}$	$\mathbb{AA}_{\mathcal{B}}$	$\mathbb{BB}_{\mathcal{AB}}$	—	—	—	$\mathbb{AB}_{\mathcal{BA}}$	$\bullet$	-1	$-1/2$
	$\mathtt{MZ}$	—	$\mathbb{BB}_{\mathcal{AB}}$	$\mathbb{AB}_{\mathcal{AB}}$	—	$\mathbb{BB}_{\mathcal{B}}$	—	$\bullet$	-1	$-1/2$
	$\mathtt{MN}$	—	$\mathbb{BB}_{\mathcal{AB}}$	$\mathbb{AB}_{\mathcal{BA}}$	—	$\mathbb{BB}_{\mathcal{A}}$	—	$\bullet$	-1	$-1/2$
	$\mathtt{MM}$	—	$2\times\mathbb{BB}_{\mathcal{AB}}$	—	—	$\mathbb{BB}_{\mathcal{A}}+\mathbb{BB}_{\mathcal{B}}$	—	—	-1	$-1/2$

	id	sources			resultants				$\Delta^{\lambda}_{\mathrm{DCJ}}$	scr
$\mathcal{M}$	$\mathtt{ZZ\underline{W}\overline{M}}$	$\mathbb{AA}_{\mathcal{B}}$	$\mathbb{BB}_{\mathcal{A}}$	$2 \times \mathbb{AB}_{\mathcal{AB}}$	—	—	—	$4 \times \bullet$	-2	$-1/2$
	$\mathtt{NN\overline{W}\underline{M}}$	$\mathbb{AA}_{\mathcal{A}}$	$\mathbb{BB}_{\mathcal{B}}$	$2 \times \mathbb{AB}_{\mathcal{BA}}$	—	—	—	$4 \times \bullet$	-2	$-1/2$
$\mathcal{N}$	$\mathtt{Z\underline{W}\overline{M}}$	$\mathbb{AA}_{\mathcal{B}}$	$\mathbb{BB}_{\mathcal{A}}$	$\mathbb{AB}_{\mathcal{AB}}$	—	—	$\mathbb{AB}_{\mathcal{BA}}$	$2 \times \bullet$	-1	$-1/3$
	$\mathtt{ZZ\underline{W}}$	$\mathbb{AA}_{\mathcal{B}}$	—	$2 \times \mathbb{AB}_{\mathcal{AB}}$	$\mathbb{AA}_{\mathcal{A}}$	—	—	$2 \times \bullet$	-1	$-1/3$
	$\mathtt{ZZ\overline{M}}$	—	$\mathbb{BB}_{\mathcal{A}}$	$2 \times \mathbb{AB}_{\mathcal{AB}}$	—	$\mathbb{BB}_{\mathcal{B}}$	—	$2 \times \bullet$	-1	$-1/3$
	$\mathtt{N\overline{W}\underline{M}}$	$\mathbb{AA}_{\mathcal{A}}$	$\mathbb{BB}_{\mathcal{B}}$	$\mathbb{AB}_{\mathcal{BA}}$	—	—	$\mathbb{AB}_{\mathcal{AB}}$	$2 \times \bullet$	-1	$-1/3$
	$\mathtt{NN\overline{W}}$	$\mathbb{AA}_{\mathcal{A}}$	—	$2 \times \mathbb{AB}_{\mathcal{BA}}$	$\mathbb{AA}_{\mathcal{B}}$	—	—	$2 \times \bullet$	-1	$-1/3$
	$\mathtt{NN\underline{M}}$	—	$\mathbb{BB}_{\mathcal{B}}$	$2 \times \mathbb{AB}_{\mathcal{BA}}$	—	$\mathbb{BB}_{\mathcal{A}}$	—	$2 \times \bullet$	-1	$-1/3$

Sources:

$\mathtt{W} : \mathbb{AA}_{\mathcal{AB}}$

$\mathtt{\overline{W}} : \mathbb{AA}_{\mathcal{A}}$

$\mathtt{\underline{W}} : \mathbb{AA}_{\mathcal{B}}$

$\mathtt{M} : \mathbb{BB}_{\mathcal{AB}}$

$\mathtt{\overline{M}} : \mathbb{BB}_{\mathcal{A}}$

$\mathtt{\underline{M}} : \mathbb{BB}_{\mathcal{B}}$

$\mathtt{Z} : \mathbb{AB}_{\mathcal{AB}}$

$\mathtt{N} : \mathbb{AB}_{\mathcal{BA}}$

DCJ-indel distance formula:

$$\mathrm{d}^{\mathrm{ID}}_{\mathrm{DCJ}}(\mathbb{A}, \mathbb{B}) = n - |\mathcal{C}| - \frac{|\mathcal{P}_{\mathbb{AB}}|}{2} + \sum_{C \in RG} \lambda(C) - \delta,$$

where δ is the value obtained by optimizing deducting path recombinations:

$$\delta = 2\mathcal{P} + 3\mathcal{Q} + 2\mathcal{T} + \mathcal{S} + 2\mathcal{M} + \mathcal{N}$$

the values $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{T}$, $\mathcal{S}$, $\mathcal{M}$ and $\mathcal{N}$ refer to the corresponding number of chains of deducting path recombinations of each type and can be obtained by a greedy approach (simple top-down screening of the table)

Singular DCJ-indel model - summary

DCJ-indel distance: $\mathsf{d}_{\mathrm{DCJ}}^{\mathrm{ID}}(\mathbb{A},\mathbb{B}) = n - |\mathcal{C}| - \frac{|\mathcal{P}_{\mathbb{A}\mathbb{B}}|}{2} + \sum_{C \in RG} \lambda(C) - \delta,$ where δ is the value obtained by optimizing deducting path recombinations

$\mathbb{A}$ and $\mathbb{B}$ are circular: $\mathsf{d}_{\mathrm{DCJ}}^{\mathrm{ID}}(\mathbb{A},\mathbb{B}) = n - |\mathcal{C}| + \sum_{C \in RG} \lambda(C)$

Computing the distance and sorting can be done in **linear time.**

Capped relational graph

Capping is a procedure that circularizes all paths of a relational graph by adding **caps** (**artificial genes**):

- if the capping is optimal, the genomic distance is preserved
- from the capped relational diagram we can derive genomes composed only of circular chromosomes

A capping may require adjacencies between caps:

$\Gamma_{\mathbb{A}}$: represents an adjacency between caps in genome $\mathbb{A}$

$\Gamma_{\mathbb{B}}$: represents an adjacency between caps in genome $\mathbb{B}$.

Capped relational graph of canonical genomes

Optimally linking paths from $RG(\mathbb{A}, \mathbb{B})$ of canonical genomes $\mathbb{A}$ and $\mathbb{B}$ into cycles can be done as follows:

id	**paths**	**linking cycle**		$\boldsymbol{\Delta}$**n**	$\boldsymbol{\Delta}$**c**	$\boldsymbol{\Delta(2\mathbb{A}\mathbb{B})}$	$\boldsymbol{\Delta}_{\mathrm{DCJ}}$
1	$\mathbb{A}\mathbb{B}$	$(\mathbb{A}\mathbb{B})$		$+0.5$	$+1$	-0.5	0
2	$\mathbb{A}\mathbb{A} + \mathbb{B}\mathbb{B}$	$(\mathbb{A}\mathbb{A}, \mathbb{B}\mathbb{B})$		$+1$	$+1$	0	0
3	$\mathbb{A}\mathbb{A}$	$(\mathbb{A}\mathbb{A}, \Gamma_{\mathbb{B}})$	$\cup$	$+1$	$+1$	0	0
4	$\mathbb{B}\mathbb{B}$	$(\mathbb{B}\mathbb{B}, \Gamma_{\mathbb{A}})$	$\cap$	$+1$	$+1$	0	0

$$\begin{cases}\text{Closing an } \mathbb{A}\mathbb{A}\text{-path (over-represented in genome } \mathbb{A} \text{ and marked with a } \cup) \text{ requires an adjacency } \Gamma_{\mathbb{B}}. \\ \text{Closing a } \mathbb{B}\mathbb{B}\text{-path (over-represented in genome } \mathbb{B} \text{ and marked with a } \cap) \text{ requires an adjacency } \Gamma_{\mathbb{A}}.\end{cases}$$

Any capping producing linking cycles as indicated on the table above is optimal:

- The value $\Delta_{\mathrm{DCJ}} = \Delta n - \Delta c - \Delta(2\mathbb{A}\mathbb{B})$ is the DCJ-effect produced by each type of linking cycle.
- All given linking cycles have $\Delta_{\mathrm{DCJ}} = 0$, therefore they preserve the DCJ distance.

Let $\begin{cases}\kappa_{\mathbb{A}}\text{: number of linear chromosomes in } \mathbb{A} \\ \kappa_{\mathbb{B}}\text{: number of linear chromosomes in } \mathbb{B}\end{cases}$

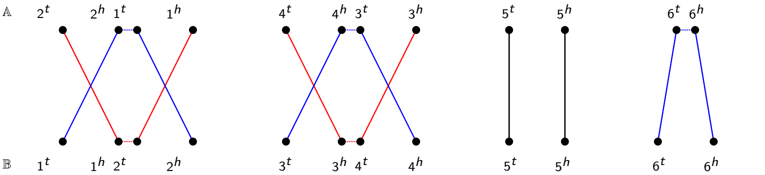
The difference between the number of $\mathbb{A}\mathbb{A}$- and of $\mathbb{B}\mathbb{B}$-paths is equal to the difference between $\kappa_{\mathbb{A}}$ and $\kappa_{\mathbb{B}}$.

An optimal capping that maximizes the number of linking cycles of type **2** minimizes the number of caps:

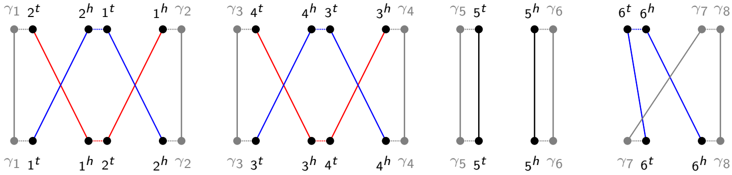
$$\begin{cases}\text{The number of caps to be added is exactly } p_* = \max\{\kappa_{\mathbb{A}}, \kappa_{\mathbb{B}}\}. \\ \text{The number of adjacencies between caps is exactly } a_* = |\kappa_{\mathbb{A}} - \kappa_{\mathbb{B}}|.\end{cases}$$

Capped relational graph of canonical genomes - example

$\mathbb{A} = [\,2\;1\,]\;\;[\,4\;3\,]\;\;[\,5\,]\;\;(\,6\,)$ and $\mathbb{B} = [\,1\;2\,]\;\;[\,3\;4\,]\;\;[\,5\,]\;\;[\,6\,]$; $p_* = 4$ and $a_* = 1$



$$\begin{aligned} \mathrm{d}_{\mathrm{DCJ}} &= n - |\mathcal{C}| - \frac{|\mathcal{P}_{\mathbb{AB}}|}{2} \\ &= 6 - 0 - 1 \\ &= 5 \end{aligned}$$



$$\begin{aligned} \mathrm{d}_{\mathrm{DCJ}} &= n + p_* - |\mathcal{C}| \\ &= 6 + 4 - 5 \\ &= 5 \end{aligned}$$

Any way of pairing the cap extremities $\gamma_1, \gamma_2, \ldots, \gamma_8$ is valid; possible derived circular genomes are:

$\mathbb{A}_\circ = (\,2\;1\;\mathtt{W}\,)\;\;(\,4\;3\;\mathtt{X}\,)\;\;(\,5\;\mathtt{Y}\,)\;\;(\,6\,)\;\;(\,\mathtt{Z}\,)$ and $\mathbb{B}_\circ = (\,1\;2\;\mathtt{W}\,)\;\;(\,3\;4\;\mathtt{X}\,)\;\;(\,5\;\mathtt{Y}\,)\;\;(\,6\;\mathtt{Z}\,)$

$(\mathtt{W}^h = \gamma_1,\ \mathtt{W}^t = \gamma_2,\ \mathtt{X}^h = \gamma_3,\ \mathtt{X}^t = \gamma_4,\ \mathtt{Y}^h = \gamma_5,\ \mathtt{Y}^t = \gamma_6,\ \mathtt{Z}^h = \gamma_7,\ \mathtt{Z}^t = \gamma_8)$

or

$\mathbb{A}_\circ = (\,2\;1\;\mathtt{W}\;4\;3\;\mathtt{X}\;5\;\;\mathtt{Y}\;\mathtt{Z}\,)\;\;(\,6\,)$ and $\mathbb{B}_\circ = (\,1\;2\;\mathtt{W}\;3\;4\;\mathtt{X}\;5\;\mathtt{Y}\;6\;\mathtt{Z}\,)$

$(\mathtt{W}^h = \gamma_3,\ \mathtt{W}^t = \gamma_2,\ \mathtt{X}^h = \gamma_5,\ \mathtt{X}^t = \gamma_4,\ \mathtt{Y}^h = \gamma_7,\ \mathtt{Y}^t = \gamma_6,\ \mathtt{Z}^h = \gamma_1,\ \mathtt{Z}^t = \gamma_8)$

Capping the relational graph - singular genomes

The sources of each chain of deducting recombinations must be properly linked together into a single cycle.

$$\begin{cases} \text{Unbalanced chains over-represented in genome } \mathbb{A} \text{ are marked with a } \cup \\ \quad \mathbb{BB}_{\varepsilon} \prec \Gamma_{\mathbb{B}}\text{: a path } \mathbb{BB}_{\varepsilon} \text{ is preferred to close a } \cup\text{-unbalanced chain; if it does not exist, an adjacency } \Gamma_{\mathbb{B}} \text{ is used} \\ \text{Unbalanced chains over-represented in genome } \mathbb{B} \text{ are marked with a } \cap \\ \quad \mathbb{AA}_{\varepsilon} \prec \Gamma_{\mathbb{A}}\text{: a path } \mathbb{AA}_{\varepsilon} \text{ is preferred to close a } \cap\text{-unbalanced chain; if it does not exist, an adjacency } \Gamma_{\mathbb{A}} \text{ is used} \end{cases}$$

In order to give the correct order of linking $\begin{cases} \text{a path } \mathbb{AB}_{\mathcal{AB}} \text{ can be represented by } \mathbb{BA}_{\mathcal{BA}} \\ \text{a path } \mathbb{AB}_{\mathcal{BA}} \text{ can be represented by } \mathbb{BA}_{\mathcal{AB}} \end{cases}$

	id	sources	linking cycle		Δn	Δc	$\Delta(2\mathbb{AB})$	$\Delta\lambda$	$\Delta^{\lambda}_{\text{DCJ}}$
$\mathcal{P}$	WM	$\mathbb{AA}_{\mathcal{AB}} + \mathbb{BB}_{\mathcal{AB}}$	$(\mathbb{AA}_{\mathcal{AB}}, \mathbb{BB}_{\mathcal{BA}})$		$+1$	$+1$	0	-2	-2
$\mathcal{Q}$	$\mathtt{WW\bar{M}\underline{M}}$	$2 \times \mathbb{AA}_{\mathcal{AB}} + \mathbb{BB}_{\mathcal{A}} + \mathbb{BB}_{\mathcal{B}}$	$(\mathbb{AA}_{\mathcal{AB}}, \mathbb{BB}_{\mathcal{B}}, \mathbb{AA}_{\mathcal{BA}}, \mathbb{BB}_{\mathcal{A}})$		$+2$	$+1$	0	-4	-3
	$\mathtt{MM\bar{W}\underline{W}}$	$2 \times \mathbb{BB}_{\mathcal{AB}} + \mathbb{AA}_{\mathcal{A}} + \mathbb{AA}_{\mathcal{B}}$	$(\mathbb{BB}_{\mathcal{AB}}, \mathbb{AA}_{\mathcal{B}}, \mathbb{BB}_{\mathcal{BA}}, \mathbb{AA}_{\mathcal{A}})$		$+2$	$+1$	0	-4	-3
$\mathcal{T}$	$\mathtt{WZ\bar{M}}$	$\mathbb{AA}_{\mathcal{AB}} + \mathbb{BB}_{\mathcal{A}} + \mathbb{AB}_{\mathcal{AB}}$	$(\mathbb{AB}_{\mathcal{AB}}, \mathbb{AA}_{\mathcal{BA}}, \mathbb{BB}_{\mathcal{A}})$		$+1.5$	$+1$	-0.5	-3	-2
	$\mathtt{WW\bar{M}}$	$2 \times \mathbb{AA}_{\mathcal{AB}} + \mathbb{BB}_{\mathcal{A}}$	$(\mathbb{AA}_{\mathcal{BA}}, \mathbb{BB}_{\mathcal{A}}, \mathbb{AA}_{\mathcal{AB}}, \mathbb{BB}_{\varepsilon} \prec \Gamma_{\mathbb{B}})$	$\cup$	$+2$	$+1$	0	-3	-2
	$\mathtt{WN\underline{M}}$	$\mathbb{AA}_{\mathcal{AB}} + \mathbb{BB}_{\mathcal{B}} + \mathbb{AB}_{\mathcal{BA}}$	$(\mathbb{AB}_{\mathcal{BA}}, \mathbb{AA}_{\mathcal{AB}}, \mathbb{BB}_{\mathcal{B}})$		$+1.5$	$+1$	-0.5	-3	-2
	$\mathtt{WW\underline{M}}$	$2 \times \mathbb{AA}_{\mathcal{AB}} + \mathbb{BB}_{\mathcal{B}}$	$(\mathbb{AA}_{\mathcal{AB}}, \mathbb{BB}_{\mathcal{A}}, \mathbb{AA}_{\mathcal{AB}}, \mathbb{BB}_{\varepsilon} \prec \Gamma_{\mathbb{B}})$	$\cup$	$+2$	$+1$	0	-3	-2
	$\mathtt{MN\bar{W}}$	$\mathbb{BB}_{\mathcal{AB}} + \mathbb{AA}_{\mathcal{A}} + \mathbb{AB}_{\mathcal{BA}}$	$(\mathbb{AB}_{\mathcal{BA}}, \mathbb{AA}_{\mathcal{A}}, \mathbb{BB}_{\mathcal{AB}})$		$+1.5$	$+1$	-0.5	-3	-2
	$\mathtt{MM\bar{W}}$	$2 \times \mathbb{BB}_{\mathcal{AB}} + \mathbb{AA}_{\mathcal{A}}$	$(\mathbb{BB}_{\mathcal{BA}}, \mathbb{AA}_{\mathcal{A}}, \mathbb{BB}_{\mathcal{AB}}, \mathbb{AA}_{\varepsilon} \prec \Gamma_{\mathbb{A}})$	$\cap$	$+2$	$+1$	0	-3	-2
	$\mathtt{MZ\underline{W}}$	$\mathbb{BB}_{\mathcal{AB}} + \mathbb{AA}_{\mathcal{B}} + \mathbb{AB}_{\mathcal{AB}}$	$(\mathbb{AB}_{\mathcal{AB}}, \mathbb{AA}_{\mathcal{B}}, \mathbb{BB}_{\mathcal{BA}})$		$+1.5$	$+1$	-0.5	-3	-2
	$\mathtt{MM\underline{W}}$	$2 \times \mathbb{BB}_{\mathcal{AB}} + \mathbb{AA}_{\mathcal{B}}$	$(\mathbb{BB}_{\mathcal{AB}}, \mathbb{AA}_{\mathcal{B}}, \mathbb{BB}_{\mathcal{BA}}, \mathbb{AA}_{\varepsilon} \prec \Gamma_{\mathbb{A}})$	$\cap$	$+2$	$+1$	0	-3	-2

	id	sources	linking cycle		Δn	Δc	$\Delta(2\mathbb{AB})$	$\Delta\lambda$	$\Delta^{\lambda}_{\text{DCJ}}$
$\mathcal{S}$	$\mathtt{ZN}$	$\mathbb{AB}_{\mathcal{AB}} + \mathbb{AB}_{\mathcal{BA}}$	$(\mathbb{AB}_{\mathcal{AB}}, \mathbb{AB}_{\mathcal{BA}})$		$+1$	$+1$	-1	-2	-1
	$\overline{\mathtt{WM}}$	$\mathbb{AA}_{\mathcal{A}} + \mathbb{BB}_{\mathcal{A}}$	$(\mathbb{AA}_{\mathcal{A}}, \mathbb{BB}_{\mathcal{A}})$		$+1$	$+1$	0	-1	-1
	$\underline{\mathtt{WM}}$	$\mathbb{AA}_{\mathcal{B}} + \mathbb{BB}_{\mathcal{B}}$	$(\mathbb{AA}_{\mathcal{B}}, \mathbb{BB}_{\mathcal{B}})$		$+1$	$+1$	0	-1	-1
	$\mathtt{W}\overline{\mathtt{M}}$	$\mathbb{AA}_{\mathcal{AB}} + \mathbb{BB}_{\mathcal{A}}$	$(\mathbb{AA}_{\mathcal{BA}}, \mathbb{BB}_{\mathcal{A}})$		$+1$	$+1$	0	-1	-1
	$\mathtt{W}\underline{\mathtt{M}}$	$\mathbb{AA}_{\mathcal{AB}} + \mathbb{BB}_{\mathcal{B}}$	$(\mathbb{AA}_{\mathcal{AB}}, \mathbb{BB}_{\mathcal{B}})$		$+1$	$+1$	0	-1	-1
	$\mathtt{WZ}$	$\mathbb{AA}_{\mathcal{AB}} + \mathbb{AB}_{\mathcal{AB}}$	$(\mathbb{AA}_{\mathcal{BA}}, \mathbb{BB}_{\varepsilon} \prec \Gamma_{\mathbb{B}}, \mathbb{AB}_{\mathcal{AB}})$	$\cup$	$+1.5$	$+1$	-0.5	-2	-1
	$\mathtt{WN}$	$\mathbb{AA}_{\mathcal{AB}} + \mathbb{AB}_{\mathcal{BA}}$	$(\mathbb{AA}_{\mathcal{AB}}, \mathbb{BB}_{\varepsilon} \prec \Gamma_{\mathbb{B}}, \mathbb{AB}_{\mathcal{BA}})$	$\cup$	$+1.5$	$+1$	-0.5	-2	-1
	$\mathtt{WW}$	$\mathbb{AA}_{\mathcal{AB}} + \mathbb{AA}_{\mathcal{AB}}$	$(\mathbb{AA}_{\mathcal{AB}}, \mathbb{BB}_{\varepsilon} \prec \Gamma_{\mathbb{B}}, \mathbb{AA}_{\mathcal{BA}}, \mathbb{BB}_{\varepsilon} \prec \Gamma_{\mathbb{B}})$	$\cup$	$+2$	$+1$	0	-2	-1
	$\mathtt{M}\overline{\mathtt{W}}$	$\mathbb{BB}_{\mathcal{AB}} + \mathbb{AA}_{\mathcal{A}}$	$(\mathbb{AA}_{\mathcal{A}}, \mathbb{BB}_{\mathcal{AB}})$		$+1$	$+1$	0	-1	-1
	$\mathtt{M}\underline{\mathtt{W}}$	$\mathbb{BB}_{\mathcal{AB}} + \mathbb{AA}_{\mathcal{B}}$	$(\mathbb{AA}_{\mathcal{B}}, \mathbb{BB}_{\mathcal{BA}})$		$+1$	$+1$	0	-1	-1
	$\mathtt{MZ}$	$\mathbb{BB}_{\mathcal{AB}} + \mathbb{AB}_{\mathcal{AB}}$	$(\mathbb{BB}_{\mathcal{BA}}, \mathbb{AB}_{\mathcal{AB}}, \mathbb{AA}_{\varepsilon} \prec \Gamma_{\mathbb{A}})$	$\cap$	$+1.5$	$+1$	-0.5	-2	-1
	$\mathtt{MN}$	$\mathbb{BB}_{\mathcal{AB}} + \mathbb{AB}_{\mathcal{BA}}$	$(\mathbb{BB}_{\mathcal{AB}}, \mathbb{AB}_{\mathcal{BA}}, \mathbb{AA}_{\varepsilon} \prec \Gamma_{\mathbb{A}})$	$\cap$	$+1.5$	$+1$	-0.5	-2	-1
	$\mathtt{MM}$	$\mathbb{BB}_{\mathcal{AB}} + \mathbb{BB}_{\mathcal{AB}}$	$(\mathbb{BB}_{\mathcal{AB}}, \mathbb{AA}_{\varepsilon} \prec \Gamma_{\mathbb{A}}, \mathbb{BB}_{\mathcal{BA}}, \mathbb{AA}_{\varepsilon} \prec \Gamma_{\mathbb{A}})$	$\cap$	$+2$	$+1$	0	-2	-1
$\mathcal{M}$	$\mathtt{ZZ}\underline{\mathtt{W}}\overline{\mathtt{M}}$	$2 \times \mathbb{AB}_{\mathcal{AB}} + \mathbb{AA}_{\mathcal{B}} + \mathbb{BB}_{\mathcal{A}}$	$(\mathbb{AB}_{\mathcal{AB}}, \mathbb{AA}_{\mathcal{B}}, \mathbb{BA}_{\mathcal{BA}}, \mathbb{BB}_{\mathcal{A}})$		$+2$	$+1$	-1	-4	-2
	$\mathtt{NN}\overline{\mathtt{W}}\underline{\mathtt{M}}$	$2 \times \mathbb{AB}_{\mathcal{BA}} + \mathbb{AA}_{\mathcal{A}} + \mathbb{BB}_{\mathcal{B}}$	$(\mathbb{AB}_{\mathcal{BA}}, \mathbb{AA}_{\mathcal{A}}, \mathbb{BA}_{\mathcal{AB}}, \mathbb{BB}_{\mathcal{B}})$		$+2$	$+1$	-1	-4	-2
$\mathcal{N}$	$\mathtt{Z}\underline{\mathtt{W}}\overline{\mathtt{M}}$	$\mathbb{AB}_{\mathcal{AB}} + \mathbb{AA}_{\mathcal{B}} + \mathbb{BB}_{\mathcal{A}}$	$(\mathbb{AB}_{\mathcal{AB}}, \mathbb{AA}_{\mathcal{B}}, \mathbb{BB}_{\mathcal{A}})$		$+1.5$	$+1$	-0.5	-2	-1
	$\mathtt{ZZ}\underline{\mathtt{W}}$	$2 \times \mathbb{AB}_{\mathcal{AB}} + \mathbb{AA}_{\mathcal{B}}$	$(\mathbb{AB}_{\mathcal{AB}}, \mathbb{AA}_{\mathcal{B}}, \mathbb{BA}_{\mathcal{BA}}, \mathbb{BB}_{\varepsilon} \prec \Gamma_{\mathbb{B}})$	$\cup$	$+2$	$+1$	-1	-3	-1
	$\mathtt{ZZ}\overline{\mathtt{M}}$	$2 \times \mathbb{AB}_{\mathcal{AB}} + \mathbb{BB}_{\mathcal{A}}$	$(\mathbb{BA}_{\mathcal{BA}}, \mathbb{BB}_{\mathcal{A}}, \mathbb{AB}_{\mathcal{AB}}, \mathbb{AA}_{\varepsilon} \prec \Gamma_{\mathbb{A}})$	$\cap$	$+2$	$+1$	-1	-3	-1
	$\mathtt{N}\overline{\mathtt{W}}\underline{\mathtt{M}}$	$\mathbb{AB}_{\mathcal{BA}} + \mathbb{AA}_{\mathcal{A}} + \mathbb{BB}_{\mathcal{B}}$	$(\mathbb{AB}_{\mathcal{BA}}, \mathbb{AA}_{\mathcal{A}}, \mathbb{BB}_{\mathcal{B}})$		$+1.5$	$+1$	-0.5	-2	-1
	$\mathtt{NN}\overline{\mathtt{W}}$	$2 \times \mathbb{AB}_{\mathcal{BA}} + \mathbb{AA}_{\mathcal{A}}$	$(\mathbb{AB}_{\mathcal{BA}}, \mathbb{AA}_{\mathcal{A}}, \mathbb{BA}_{\mathcal{AB}}, \mathbb{BB}_{\varepsilon} \prec \Gamma_{\mathbb{B}})$	$\cup$	$+2$	$+1$	-1	-3	-1
	$\mathtt{NN}\underline{\mathtt{M}}$	$2 \times \mathbb{AB}_{\mathcal{BA}} + \mathbb{BB}_{\mathcal{B}}$	$(\mathbb{BA}_{\mathcal{AB}}, \mathbb{BB}_{\mathcal{B}}, \mathbb{AB}_{\mathcal{BA}}, \mathbb{AA}_{\varepsilon} \prec \Gamma_{\mathbb{A}})$	$\cap$	$+2$	$+1$	-1	-3	-1

	remaining paths	linking cycle		Δn	Δc	$\Delta(2\mathbb{AB})$	$\Delta\lambda$	$\Delta^{\lambda}_{\mathrm{DCJ}}$
1	$\mathbb{AB}_*$	$(\mathbb{AB}_*)$		$+0.5$	$+1$	-0.5	0	0
2	$\mathbb{AA}_* + \mathbb{BB}_*$	$(\mathbb{AA}_*, \mathbb{BB}_*)$		$+1$	$+1$	0	0	0
3	$\mathbb{AA}_*$	$(\mathbb{AA}_*, \Gamma_{\mathbb{B}})$	$\cup$	$+1$	$+1$	0	0	0
4	$\mathbb{BB}_*$	$(\mathbb{BB}_*, \Gamma_{\mathbb{A}})$	$\cap$	$+1$	$+1$	0	0	0

Any capping producing linking cycles following a top-down screening of the table above is optimal:

- $\Delta^{\lambda}_{\mathrm{DCJ}} = \Delta n - \Delta c - \Delta(2\mathbb{AB}) + \Delta\lambda$ gives the DCJ-indel-effect produced by each type of linking cycle.
- All given linking cycles have $\Delta^{\lambda}_{\mathrm{DCJ}}$ equivalent to the respective chain of deducting recombinations, therefore they achieve the optimal DCJ-indel distance.

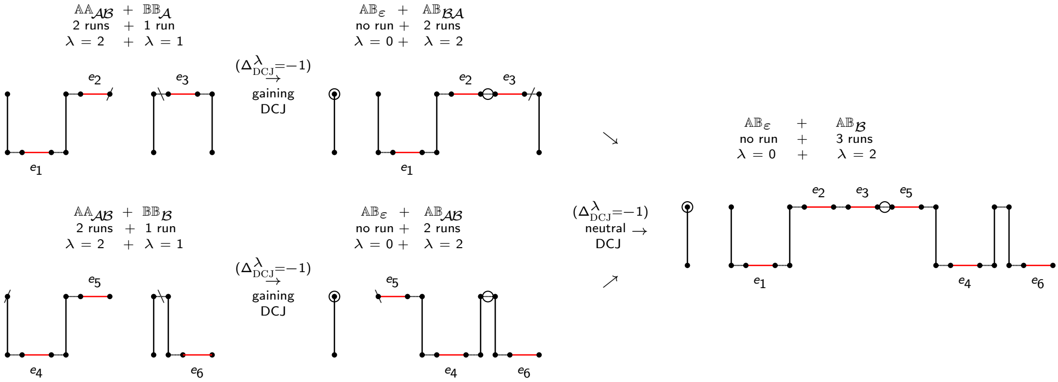
P1: After identifying chains of recombinations
- either there are no unbalanced chains
- or there are only $\cup$-unbalanced chains (over-repr. in $\mathbb{A}$)
- or there are only $\cap$-unbalanced chains (over-repr. in $\mathbb{B}$)

P2: When an unbalanced chain is being linked
- if there is a remaining indel-free $\mathbb{AA}_\varepsilon/\mathbb{BB}_\varepsilon$ (of the under-repr. genome), it is used to link the chain
- otherwise there is no remaining $\mathbb{AA}_*/\mathbb{BB}_*$ (of the under-repr. genome) and an adjacency $\Gamma_{\mathbb{A}/\mathbb{B}}$ links the chain

Any optimal capping that links all possible chains of deducting recombinations as described above and, for the remaining paths, maximizes the number of linking cycles of type **2** minimizes the number of caps:
- The number of caps to be added is exactly $p_* = \max\{\kappa_{\mathbb{A}}, \kappa_{\mathbb{B}}\}$.
- The number of adjacencies between caps is exactly $a_* = |\kappa_{\mathbb{A}} - \kappa_{\mathbb{B}}|$.

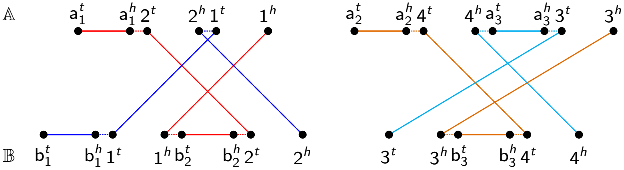
Capped relational graph of singular genomes - example

$$\text{Deducting chain of path recombinations} \begin{cases} \text{transforming} & 2 \times \mathbb{AA}_{\mathcal{AB}} + \mathbb{BB}_{\mathcal{A}} + \mathbb{BB}_{\mathcal{B}} \\ \text{into} & 3 \times \mathbb{AB}_{\varepsilon} + \mathbb{AB}_{\mathcal{B}} \\ \text{with} & \text{overall } \Delta^{\lambda}_{\mathrm{DCJ}} = -3 \end{cases}$$



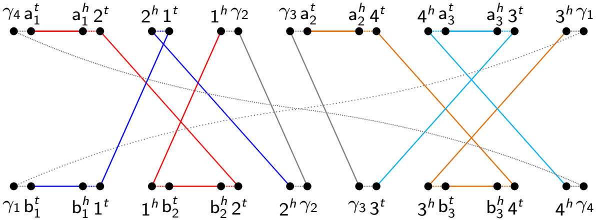
Capped relational graph of singular genomes - example

$\mathbb{A} = [\mathrm{a}_1\ 2\ 1]\ \ [\mathrm{a}_2\ 4\ \mathrm{a}_3\ 3]$ and $\mathbb{B} = [\mathrm{b}_1\ 1\ \mathrm{b}_2\ 2]\ \ [3\ \mathrm{b}_3\ 4]$; $p_* = 2$ and $a_* = 0$



Components: $2 \times \mathbb{AA}_{\mathcal{AB}}$, $\mathbb{BB}_{\mathcal{A}}$, $\mathbb{BB}_{\mathcal{B}}$

$$
\begin{aligned}
\mathrm{d}_{\mathrm{DCJ}}^{\mathrm{ID}} &= n - |\mathcal{C}| - \frac{|\mathcal{P}_{\mathbb{AB}}|}{2} + \sum \lambda(C) - \delta \\
&= 4 - 0 - 0 + 6 - 3 \\
&= 7
\end{aligned}
$$



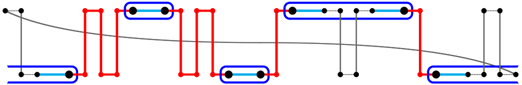
Linking cycle: $(\mathbb{AA}_{\mathcal{AB}}, \mathbb{BB}_{\mathcal{B}}, \mathbb{AA}_{\mathcal{BA}}, \mathbb{BB}_{\mathcal{A}})$

$$
\begin{aligned}
\mathrm{d}_{\mathrm{DCJ}}^{\mathrm{ID}} &= n + p_* - |\mathcal{C}| + \sum \lambda(C) \\
&= 4 + 2 - 1 + 2 \\
&= 7
\end{aligned}
$$

The four sources of a chain of deducting recombinations are optimally linked into a single cycle.

Indel-potential via transitions

One indel-enclosing cycle:



$\aleph = \Lambda = 4$

indel edges , **runs** , **transitions**

$\Lambda(C)$ is the number of **runs** in cycle C

$\aleph(C)$ is the number of **transitions** in cycle C

Λ	$\aleph$	r	λ	
0	0	0	0	cycles
1	0	1	1	cycles and singletons
2	2	1	2	cycles
4	4	1	3	cycles
6	6	1	4	cycles
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Indel-potential of a component C:

$$\lambda(C) = \begin{cases} 0 & \text{if } \Lambda(C) = 0 \ (C \text{ is indel-free}) \\ 1 & \text{if } \Lambda(C) = 1 \\ \frac{\Lambda(C)}{2} + 1 & \text{if } \Lambda(C) \geq 2 \end{cases}$$

$$\lambda(C) = \frac{\aleph(C)}{2} + r(C)$$

$$r(C) = \begin{cases} 1, & \text{component } C \text{ is indel-enclosing} \\ 0, & \text{component } C \text{ is indel-free} \end{cases}$$

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